Heterotic string theory interrelations

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Abstract

We establish a symmetry map which relates two low-energy heterotic string theories with different numbers of the Abelian gauge fields compactified from the diverse to three dimensions on a torus. We discuss two applications of the established symmetry: a generation of the heterotic string theory solutions from the stationary Einstein-Maxwell fields and one non-trivial submersion of the heterotic string theory into the bosonic one.

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1 Introduction

Symmetries play a crucial role in the study of superstring theory [1], [2], [3], [4], [5]. In this study the first step is related to the search of hidden symmetries of the different field theory limits of superstring theory. After that it becomes possible to make the second step - to generalize the obtained results to the case of exact superstring theory by exploring of the supersymmetry arguments [6], [7], [8].

The heterotic string theory becomes a field theory of its massless modes at low energies. The bosonic sector of this field theory leads to the symmetric space model coupled to gravity after the toroidal compactification to three dimensions [3], [5]. This is the symmetric space model with the null-curvature matrix parameterizing the coset $O(d+1, d+1+n)/O(d+1) \times O(d+1+n)$, where d is the number of compactified dimensions and n is the number of Abelian gauge fields in the multidimensional theory. In fact one has a series of the low-energy heterotic string theory systems labeled by the numbers d and n, and all these field theories are of the interest in framework of the basic exact heterotic string theory activity. Of course, the special cases of $d = 7, n = 16$ (the consistent heterotic string theory) and $d = 23, n = 0$ (the consistent bosonic string theory) are the most physically important. However, some supergravities and the general superstring theory web of dualities give a sufficient motivation for the study of whole effective set of the (d, n)-labeled heterotic string theories.

In this paper we establish a symmetry map which relates two different representatives of the (d, n)-series of the theories. This map is the on-shell symmetry, so one can use it also as some new symmetry extension of the simpler theory to the more complicated one (as a rule, in concrete work the simpler theory is characterized by the smaller values of d and n). Below we show how one can extend the solution space of the stationary Einstein-Maxwell theory [9] to the one of heterotic string theory with the arbitrary numbers d and n. This result opens a new and actually promising way for generation of the heterotic string theory solutions (see [10] for different examples of generation starting from the General Relativity). Another interesting application of the established symmetry map is related to some simple submersion of the heterotic string theory into the bosonic one. This submersion seems surprising in framework of the conventional subsystem search for the following generation procedure. Actually, in view of this submersion one generates the solutions of the theory without Abelian fields from the solutions of the theory with non-trivial Abelian gauge sector. Also it seems physically interesting that the critical case of the heterotic string theory exactly transforms to the critical bosonic string theory one in framework of the simplest new map realization.

2 Symmetric space model

In [11] it was shown that the low-energy heterotic string theory toroidally compactified to three dimensions can be represented in terms of the single real $(d+1) \times (d+1+n)$ matrix potential $\mathcal{Z}$ coupled to the effective 3-dimensional metric $h_{\mu\nu}$. The corresponding action reads:

$$\begin{aligned} S_3 &= \int d^3x h^{1/2} \left(-R_3 + L_3\right), \\ L_3 &= \mathrm{Tr}\left[\nabla\mathcal{Z}\left(\Xi - \mathcal{Z}^T\Sigma\mathcal{Z}\right)^{-1}\nabla\mathcal{Z}^T\left(\Sigma - \mathcal{Z}\Xi\mathcal{Z}^T\right)^{-1}\right], \end{aligned} \tag{1}$$

where Σ and Ξ are the $(d+1)\times(d+1)$ and $(d+1+n)\times(d+1+n)$ matrices respectively of the form $\mathrm{diag}(-1,-1;1,...,1)$ (see [11] for the details). It is important to note, that the three remaining coordinates x^μ are Euclidean, the time dimension is also compactified together with the 'true' extra dimensions. Another important fact is that the potential $\mathcal{Z}$ realizes the matrix representation of the lowest possible matrix dimensionality: the symmetric space model $O(d+1, d+1+n)/O(d+1) \times O(d+1+n)$ has exactly $(d+1) \times (d+1+n)$ degrees of freedom. Then, from Eq. (1) it immediately follows that the transformation

$$\mathcal{Z} \rightarrow \mathcal{C}_L\mathcal{Z}\mathcal{C}_R \tag{2}$$

is a symmetry if

$$\mathcal{C}_L\Sigma\mathcal{C}_L = \Sigma, \quad \mathcal{C}_R\Xi\mathcal{C}_R = \Xi, \tag{3}$$

i.e., if $\mathcal{C}_L \in O(2, d-1)$ and $\mathcal{C}_R \in O(2, d-1+n)$. In [12] it was shown that the group $O(2, d-1) \times O(2, d-1+n)$ forms the total subgroup of charging symmetries for the theory under consideration. This subgroup preserves the trivial values of the spatial asymptotics of the all three-dimensional fields (i.e. the value $\mathcal{Z} = 0$). These symmetries generalize the subgroup of the Einstein-Maxwell charging symmetries [9] to the heterotic string theory case. It is important to note, that the reconstruction of the multidimensional fields from the asymptotically flat three-dimensional solution can lead to NUT, magnetic and other kind of Dirac string peculiarities. In all other senses such multidimensional solution is also asymptotically trivial. The asymptotically flat solutions of heterotic string theory play a crucial role in the black hole physics [13] and in the other physical applications of the superstring theory.

The $\mathcal{Z}$-formulation is closely related to the null-curvature matrix approach originally developed in [5]. Here the history is the following: in [14] it was constructed some new null-curvature matrix which is closely related to the Ernst matrix potential approach. This approach establishes the explicit analogy between the Einstein-Maxwell and heterotic string theories (see the conventional Ernst potential representation of the stationary Einstein-Maxwell theory in [15]). In [12] the potential $\mathcal{Z}$ was initially introduced for the simple linearization of action of the heterotic string theory charging symmetries. After that in [11] the $\mathcal{Z}$-formalism had been developed in details as the new and powerful consistent approach. In this article we give, may be, the most important application of this new approach to the physically and mathematically interesting problems of the low-energy heterotic string theory.

3 New symmetry map

The new symmetry map naturally arises in framework of one simple anzats consideration. Let us start with

$$\mathcal{Z} = \xi_{rl} L_l R_r^T, \tag{4}$$

where $l = 1, ..., \mathcal{N}_l$, $r = 1, ..., \mathcal{N}_r$ $\quad \xi_{rl} = \xi_{rl}(x^\mu)$ is the set of $\mathcal{N}_l \times \mathcal{N}_r$ dynamical functions and L_l, R_r are the constant columns of the dimensions $(d+1) \times 1$ and $(d+1+n) \times 1$ respectively. The question is when the dynamical equations

$$\begin{aligned} &\nabla^2 \mathcal{Z} + 2\nabla \mathcal{Z} \Xi \mathcal{Z}^T \left(\Sigma - \mathcal{Z}\Xi\mathcal{Z}^T\right)^{-1} \nabla \mathcal{Z} = 0, \\ &R_{3\,\mu\nu} = \mathrm{Tr}\left[\mathcal{Z}_{,(\mu} \left(\Xi - \mathcal{Z}^T \Sigma \mathcal{Z}\right)^{-1} \mathcal{Z}^T_{,\nu)} \left(\Sigma - \mathcal{Z}\Xi\mathcal{Z}^T\right)^{-1}\right], \end{aligned} \tag{5}$$

which correspond to the action (1), are automatically satisfied in framework of the structure (4). To answer on this question, one needs in substitution of Eq. (4) to Eq. (5); this work can be elegantly performed in terms of the operators

$$\Pi^{(L)}_{l_1 l_2} = L_{l_1} L^T_{l_2} \Sigma, \quad \Pi^{(R)}_{r_1 r_2} = R_{r_1} R^T_{r_2} \Xi \tag{6}$$

and the related constants

$$\kappa^{(L)}_{l_1 l_2} = L^T_{l_1} \Sigma L_{l_2}, \quad \kappa^{(R)}_{r_1 r_2} = R^T_{r_1} \Xi R_{r_2}, \tag{7}$$

which are also related by the following multiplication table:

$$\Pi^{(L)}_{l_1 l_2} \Pi^{(L)}_{l_3 l_4} = \kappa^{(L)}_{l_2 l_3} \Pi^{(L)}_{l_1 l_4}, \quad \Pi^{(R)}_{r_1 r_2} \Pi^{(R)}_{r_3 r_4} = \kappa^{(R)}_{r_2 r_3} \Pi^{(R)}_{r_1 r_4}. \tag{8}$$

These relations characterize the operators (6) as projectors. In view of this fact our approach can be naturally named as 'projective'; this name will be also supported by the following consideration and by the main result. Let us now combine the quantities $\xi_{rl}, \kappa^{(L)}_{l_1l_2}$ and $\kappa^{(R)}_{r_1r_2}$ into the matrices ξ, $\kappa^{(L)}$ and $\kappa^{(R)}$ respectively. Then the straightforward calculation show that the motion equations (5) become automatically satisfied if

$$\nabla^2\xi + 2\nabla\xi\kappa^{(R)}\xi^T\left([\kappa^{(L)}]^{-1} - \xi\kappa^{(R)}\xi^T\right)^{-1}\nabla\xi = 0,$$
$$R_{3\,\mu\nu} = \mathrm{Tr}\left[\xi_{,(\mu}\left([\kappa^{(R)}]^{-1} - \xi^T[\kappa^{(L)}]\xi\right)^{-1}\xi^T_{,\nu)}\left([\kappa^{(L)}]^{-1} - \xi\kappa^{(R)}\xi^T\right)^{-1}\right], \tag{9}$$

where it was supposed that the matrices $\kappa^{(L)}$ and $\kappa^{(R)}$ are non-degenerated.

We state that the equations (9) are of the form (5) but written in the 'random' terms. Actually, let us denote the signature matrices for $\kappa^{(L)}$ and $\kappa^{(R)}$ as $\tilde{\Sigma}$ and $\tilde{\Xi}$ respectively. Then from the corresponding algebraic theorem it follows an existence of the matrices $\aleph^{(L)}$ and $\aleph^{(R)}$ such that

$$\kappa^{(L)} = \aleph^{(L)\,T}\tilde{\Sigma}\aleph^{(L)}, \quad \kappa^{(R)} = \aleph^{(R)\,T}\tilde{\Xi}\aleph^{(R)}. \tag{10}$$

Let us also define the new dynamical field

$$\tilde{\mathcal{Z}} = \aleph^{(L)}\xi\aleph^{(R)\,T}. \tag{11}$$

Then, as it is easy to verify, from Eq. (9) it follows that

$$\nabla^2\tilde{\mathcal{Z}} + 2\nabla\tilde{\mathcal{Z}}\tilde{\Xi}\tilde{\mathcal{Z}}^T\left(\tilde{\Sigma} - \tilde{\mathcal{Z}}\tilde{\Xi}\tilde{\mathcal{Z}}^T\right)^{-1}\nabla\tilde{\mathcal{Z}} = 0,$$
$$R_{3\,\mu\nu} = \mathrm{Tr}\left[\tilde{\mathcal{Z}}_{,(\mu}\left(\tilde{\Xi} - \tilde{\mathcal{Z}}^T\tilde{\Sigma}\tilde{\mathcal{Z}}\right)^{-1}\tilde{\mathcal{Z}}^T_{,\nu)}\left(\tilde{\Sigma} - \tilde{\mathcal{Z}}\tilde{\Xi}\tilde{\mathcal{Z}}^T\right)^{-1}\right], \tag{12}$$

i.e., the system which is related to (5) by the help of substitution

$$\tilde{\mathcal{Z}} \leftrightarrow \mathcal{Z}, \quad \tilde{\Sigma} \leftrightarrow \Sigma, \quad \tilde{\Xi} \leftrightarrow \Xi. \tag{13}$$

This system can be derived from the Lagrangian

$$\tilde{L}_3 = \mathrm{Tr}\left[\nabla\tilde{\mathcal{Z}}\left(\tilde{\Xi} - \tilde{\mathcal{Z}}^T\tilde{\Sigma}\tilde{\mathcal{Z}}\right)^{-1}\tilde{\mathcal{Z}}^T\left(\tilde{\Sigma} - \tilde{\mathcal{Z}}\tilde{\Xi}\tilde{\mathcal{Z}}^T\right)^{-1}\right], \tag{14}$$

which has exactly the heterotic string theory form (1). The necessary additional element of the correspondence (13) follows from Eqs. (4), (7). To obtain it in the convenient form, let us define the new set of constant columns

$$L_{0l_1} = \left([\aleph^{(L)}]^{-1}\right)_{l_2l_1}L_{l_2}, \quad R_{0r_1} = \left([\aleph^{(R)}]^{-1}\right)_{r_2r_1}R_{r_2}. \tag{15}$$

Then from Eq. (7) one obtains that

$$L_{0l_1}^T \Sigma L_{0l_2}^T = \tilde{\Sigma}_{l_1 l_2}, \quad R_{0r_1}^T \Xi R_{0r_2}^T = \tilde{\Xi}_{r_1 r_2}. \tag{16}$$

Let us now combine the columns L_{0l} and R_{0r} to the matrices L and R respectively in the natural way (then, for example, L_{0l} is the l-th column of L). Then Eq. (16) takes the form of

$$L^T \Sigma L = \tilde{\Sigma}, \quad R^T \Xi R = \tilde{\Xi}, \tag{17}$$

whereas Eq. (4) reads:

$$\mathcal{Z} = L \tilde{\mathcal{Z}} R^T. \tag{18}$$

The equations (2), (14), (17) and (18) define the correspondence (13) completely. A simple algebraical analysis of Eq. (17) shows that $\mathcal{N}_L \leq d+1$, $\mathcal{N}_R \leq d+1+n$ and the number of -1 in $\tilde{\Sigma}$ and $\tilde{\Xi}$ is not more than 2. We would like to interpret the $\tilde{\mathcal{Z}}$-theory as some other example of low-energy heterotic string theory. To do this let us completely define our anzats by the demanding that $\mathcal{N}_L \leq \mathcal{N}_R$ and that both the matrices $\tilde{\Sigma}$ and $\tilde{\Xi}$ are of the form $\mathrm{diag}(-1,-1;1,...,1)$. Then the $\tilde{\mathcal{Z}}$-theory actually becomes the heterotic string theory with $\tilde{d} = \mathcal{N}_L - 1$ toroidally compactified dimensions and $\tilde{n} = \mathcal{N}_R - \mathcal{N}_L$ original Abelian gauge fields. It is easy to see that the correspondence (13) becomes the general map which acts in the whole (d,n)-labeled series of the low-energy heterotic string theory.

4 Some applications

Our first example is the $\tilde{\mathcal{Z}}$-theory with $\tilde{n} = 2(\tilde{d}+1)$ and the $\mathcal{Z}$-theory with $n=0$, $d = 3\tilde{d}+2$. This special case includes the critical heterotic and bosonic string theories ($\tilde{d} = 7, \tilde{n} = 16$ and $d = 23, n = 0$ respectively). We state that this special case can be considered in framework of the correspondence (13). Actually, let us take the matrices L and R as

$$L = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad R = 1, \tag{19}$$

where the block unit matrix in L has the dimension $(\tilde{d}+1) \times (\tilde{d}+1)$. Then the relation

$$\mathcal{Z} = \begin{pmatrix} \tilde{\mathcal{Z}} \\ 0 \end{pmatrix} \tag{20}$$

is the realization of Eq. (18) in this situation. Note, that Eq. (20) transform any solution of the heterotic string theory to the corresponding solution of the bosonic string theory, including the critical cases.

Our second example is the following. Let us put $\tilde{d} = 1, \tilde{n} = 2$ (this theory is interesting itself from the point of view of $D = N = 4$ supergravity, see [7] and references therein). In this case $\tilde{\mathcal{Z}}$ is the 2×4 matrix. Let us separate it to two 2×2 blocks,

$$\tilde{\mathcal{Z}} = (\tilde{\mathcal{Z}}_1, \tilde{\mathcal{Z}}_2), \tag{21}$$

and after that let us take these blocks in the following special form:

$$\tilde{\mathcal{Z}}_a = \begin{pmatrix} z_a^{'} & z_a^{''} \\ -z_a^{''} & z_a^{'} \end{pmatrix}, \tag{22}$$

where $a = 1, 2$. In fact Eq. (22) gives the 2×2 matrix representation of the complex potentials

$$z_a = z_a^{'} + i z_a^{''}; \tag{23}$$

it is easy to prove that the anzats (22) is consistent, i.e., the dynamical equations (12) do not impose any additional restrictions on the potentials z_a. These equations become the equations on the complex 1×2 matrix potential z,

$$z = (z_1, z_2), \tag{24}$$

and can be derived from the action (1) with the Lagrangian

$$L_{EM} = 2\frac{\nabla z(\sigma_3 - z^+ z)^{-1} \nabla z^+}{1 - z\sigma_3 z^+}, \tag{25}$$

where σ_3 is one of the Pauli matrices. We state that this effective theory coincides with the stationary Einstein-Maxwell one. To prove this statement, let us introduce the potentials $\mathcal{E}$ and $\mathcal{F}$, where

$$\mathcal{E} = \frac{1 - z_1}{1 + z_1}, \quad \mathcal{F} = \frac{\sqrt{2} z_2}{1 + z_1}. \tag{26}$$

Then for the Lagrangian (24) one has:

$$L_{EM} = \frac{1}{2f^2} \left| \nabla \mathcal{E} - \bar{\mathcal{F}} \nabla \mathcal{F} \right|^2 - \frac{1}{f} \left| \nabla \mathcal{F} \right|^2, \tag{27}$$

where $f = 1/2(\mathcal{E}+\bar{\mathcal{E}}-|\mathcal{F}|^2)$. It is clear that (27) is the conventional action for the stationary Einstein-Maxwell theory written in terms of the Ernst potentials $\mathcal{E}$ and $\mathcal{F}$ (see [15]). Thus, Eqs. (21)-(25) define the $\tilde{\mathcal{Z}}$-system which is equivalent to the stationary Einstein-Maxwell theory. It can be used for generation of the heterotic string theory solutions from the stationary Einstein-Maxwell ones by the help of the above established symmetry map. Of course, all these results can be easily modified to the 'cosmological' axisymmetric case when the effective three-dimensional coordinate space contains the time dimension.

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5 Conclusion

A simplicity which characterizes the process of results obtaining is related to remarkable properties of the $\mathcal{Z}$-representation of the low-energy heterotic string theory compactified to three dimensions on a torus. A very similar formalism for the stationary Einstein-Maxwell theory one can find in [16], where the equivalents of our potentials z_a had been introduced. It is important to note that the $\mathcal{Z}$-approach is especially useful in framework of the asymptotically flat solutions of the theory. In fact our symmetry map (18) coincides with the general charging symmetry transformation [12] in the case when $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ are of the same matrix dimensionality (see Eq. (2)). Moreover, in the general case of the different dimensionalities of the potentials $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ our symmetry map possessing the charging symmetry invariance property itself. This means the following: an attempt to generalize the result given by Eq. (18) by the help of the transformation (2) applied to the $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$ potentials leads to the non-important reparameterization of the constant matrices L and R. Actually, it is easy to see that this reparameterization preserves the restrictions (17), so it play the zero role if one takes the general solution of the algebraic equations (17).

At the end of this paper let us briefly discuss some natural perspectives. First of all, it is possible to 'translate' the material given in the previous section from the language of three-dimensional potentials to the form of physical field components. All the necessary information for this 'translation' can be found in [12] and [11]; in this short paper we will not give all important but technically complicated details. Second, it will be interesting to calculate the concrete heterotic string theory solutions from the known Einstein-Maxwell theory ones using our new symmetry map. For example, it is possible to construct the charging symmetry complete extension of the Kerr-Newman solution. The corresponding work is now in progress.

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